ON BRADLEY-TERRY MODELS FOR SYMMETRICAL PAIRS

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1. INTRODUCTION

In order to reduce the tedium of experiments in paired comparisons fractionation of pairs can be made. The methods of fractionation lead to an important fraction labelled symmetrical pairs. If there be t stimuli designated by $S_1, S_2, \ldots, S_t$, the symmetrical pairs are $S_1 S_2$, $S_2 S_3$, ..., S_{t-1} S_t, S_t S_1. Symmetrical pairs are very useful for sensory testing, testing preferences for industrial products, building materials, consumer protection and guidance, market research, certification marking for standardisation etc. For assessment of quality of different products at reduced cost the method is very useful. It is also useful in testing trends in treatment ratings. Some designs for fractional pairs are given in Sadasivan (1970). A model for symmetrical pairs was given in Sadasivan and Sundaram (1974). In the present paper properties of two different mathematical models for evaluation of experimental data from symmetrical pairs have been studied.

2. BRADLEY-TERRY MODEL WITHOUT TIES

2.1. The Model

Let the true ratings for the t treatments be $\pi_1, \pi_2, \ldots, \pi_t$ on a particular subjective continuum such that $\pi_i > 0$ and $\Sigma \pi_i = 1$. The probability that treatment i obtains top ranking when treatment i appears with treatment j in a block is $\pi_i/(\pi_i + \pi_j)$. Following Bradley-Terry notation and considering the appropriate likelihoods for symmetrical pairs within a repetition and for all n repetitions, we reach the likelihood function in the general form as

$$L(\underset{\sim}{\pi}) = \prod_{i=1}^{t} (\pi_i{}^{a_i} \; (\pi_i + \pi_{i+1})^{-(n)} \qquad \ldots(1)$$

where

$$a_i = 4n - \sum_{k=1}^{n} \sum_{\substack{j=i-1 \\ i+1}} r_{ijk}$$

and

$$\underset{\sim}{\pi} = (\pi_1, \pi_2, \ldots, \pi_t).$$

A general class of tests of the null hypothesis

$$H_0 : \pi_i = 1/t$$
$$(i = 1, 2, \ldots, t)$$

against

$$H_a : \pi_i = \pi(h)$$
$$(h = 1, 2, \ldots, m),$$
$$i = s_{h-1} + 1, \ldots, s_h$$

where

$$S_0 = 0,$$
$$S_m = t,$$

$$\sum_{h=1}^{m} (s_h - s_{h-1})\ \pi(h) = 1,$$

h=the index for the class in which a treatment rating falls, s_h=the cumulative number of ratings up toclass h and m=the number of classes in which the treatment ratings are equal, is made by maximising $ln\ L$ using Lagrange's multipliers and deriving the general test statistic which is a monotonic function of the likelihood ratio (*i.e.*, $ln\lambda$). The normal equations to obtain $p(h)$, the estimate of $\pi(h)$ are

$$\left.\begin{aligned} \frac{A_h}{p(h)} - \frac{n(s_h - s_{h-1} - 1)}{p(h)} \\ - \frac{n}{p(h)+p(h+1)} - \frac{n}{p(h)+p(h-1)} \\ = 0 (h = 1, 2, \ldots, m) \end{aligned}\right\} \quad \ldots(2)$$

and

$$\sum_{h=1}^{m} (s_h - s_{h-1})\ \ p(h) = 1$$

where

$$A_h = \sum_{i=s_h+1}^{s_{h+1}} \left[4n - \sum_{i=1}^{k} \sum_{\substack{j=i-1 \ \& \\ i+1}} r_{ijk} \right]$$

and the test statistic

$$B = -\sum_{h=1}^{m} A_h \log p(h) + n \sum_{h=1}^{m} (s_h - s_{h-1} - 1) \log p(h)$$

$$+ n \sum_{h=1}^{m} \log \{p(h) + p(h+1)\} + n(t-m) \log 2. \qquad \ldots(3)$$

Hence the asymptotic test statistic becomes

$$T = -2\, ln\lambda = 2nt\, ln2 - 2B\, ln\, 10.$$

where ln, log are logarithms to base e and 10 respectively.

The two special tests work out as follows :

Case 1. H_1 : no π_i is assumed equal to any π_j $(i \neq j)$. The normal equations in this case are

$$\frac{a_i}{p_i} - n \sum_{\substack{j=i-1 \ \& \\ i+1}} (p_i + p_j)^{-1} = 0 \quad (i = 1, 2, \ldots, t) \qquad \ldots(4)$$

subject to the constraint

$$\sum_{i=1}^{t} p_i = 1 \qquad \ldots(5)$$

The solution of these equations can be obtained by iteration. The test statistic becomes

$$B^{(1)} = n \sum_{i=1}^{t} \log (p_i + p_{i+1}) - \sum_{i=1}^{t} a_i \log p_i \qquad \ldots(6)$$

An asymptotic test is provided by the statistic

$$T^{(1)} = 2nt\, ln_2 - 2B^{(1)}\, ln_{10} \qquad \ldots(7)$$

Under H_0, $T(^1)$ is distributed as a central χ^2 with $(t-1)$ degree of freedom.

Case 2.

$$H_2: \pi_i=\pi(i=1, 2, ..., s),$$
$$\pi_j=(1-s\pi)/(t-s)$$
$$=\pi' \; (j=s+1; \; ... \; t).$$

The likelihood function under H_2 is

$$L(\underset{\sim}{\pi}/H_2)=\pi^{B_1} (\pi')^{B_2} 2^{-nt-2)} (\pi+\pi')^{-2n}$$

where

$$B_1=\sum_{i=1}^{s} a_i-n\,(s-1)$$

and

$$B_2=\sum_{i=s+1}^{t} a_i-n\,(t-s-1).$$

The estimation equations are

$$\frac{B_1}{p}-2n\,(p-p')^{-1}=0 \qquad ...(8)$$

$$\frac{B_2}{p}-2n\,(p+p')^{-1}=0 \qquad ...(9)$$

subject to the constraint

$$sp+(t-s)\,p'=1 \qquad ...(10)$$

where

$$p'=(1-sp)/(t-s),$$

p being an estimate of π. Then

$$p=\frac{B_1}{(2n-B_1)\,(t-s)+sB_1};$$

$$B^{(2)}=2n \log (p+p')+n(t-2) \log 2 -B_1 \log p-B_2 \log p' \qquad ...(11)$$

$$T(^2)=2n \; ln 2-2B^{(2)} \; ln\, 10$$

which under the null hypothesis is distributed as a chi-square with one degree of freedom. For exact test the distribution of $B(^2)$ should be tabled.

An estimate p of π is given by

$$p=\frac{X}{(2n-X)\,(t-s)+X_s}$$

where X is the number of times a treatment of the first group, ranks above a treatment of the second group. From the model for paired comparisons, the probability that a treatment i of the first group ranks above a treatment j of the second group in any of the n repetitions is

$$P(r_{ijk}=1)=\frac{\pi(t-s)}{1+(t-2s)}.$$

Then

$$\text{Est } P(r_{ijk}=1)=\frac{X}{2n}$$

which shows that case (2) reduces to the sign test.

2.2. Alternative Methods of Derivation of the Likelihood under Case (1)

It may be noted that the likelihood function (1) may also be set up in the following two ways :

(*a*) Consider the generalised likelihood for full pairs with n_{ij} repetitions on pair (i, j) from Dykstra (1960). Put $n_{ij}=n$ for pairs compared and zero for the others. Then

$$L(\underset{\sim}{\pi})=\prod_{i=1}^{t} \pi_i^{\,a_i}(\pi_i+\pi_{i+1})^{-n}$$

where a_i is the number of preferences for treatment i from the whole experiment.

(*b*) Start with the general likelihood for multi-variate paired comparisons with p variables and n_{ij} repetitions per pair (i, j) set up by Davidson and Bradley (1969).

To obtain the likelihood under Case (1) put

$$p=1,$$
$$\nu_{\alpha i}=a_i,$$
$$\pi_{\alpha i}=\pi_i,$$
$$n_{ij}=n_{i,\,i+1}$$
$$=n,$$
$$f(s/i,j)=a_{ij}$$

=the number of preferences for i in the comparisons of (i, j) and $lnh(S/ij)=0$, where the symbols have the same meanings as in Davidson and Bradley. Then

$$ln\ L(\underset{\sim}{\pi})=\sum_{i=1}^{t} a_i\ ln\ \pi_i-n\sum_{i=1}^{t} ln\ (\pi_i-\pi_{i+1}).$$

2.3. Tables for $B(^1)$

The $B(^1)$ function under case (1) has been tabled for a limited number of cases. For

$$t=3,$$
$$\lambda=1,$$
$$b=3,$$
$$r=2$$

and

$$k=2,$$

the Bradley-Terry tables for full pairs can be used since the $B(^1)$ function and the corresponding probabilities in this case are identical. Tables are given in the Appendix corresponding to parametric combinations for

$$t=4,$$
$$n=1, 2$$

and

$$t=5,$$
$$n=1.$$

2.4. Power Function for Symmetrical Pairs

Consider the generalized power function for multivariate paired comparisons (Davidson and Bradley, 1970). The non-centrality parameter for the distribution of the likelihood ratio statistic in our case can be obtained therefrom by appropriate substitutions as

$$\lambda_s=\frac{t^2}{4N}\sum_{i=1}^{t} n_{i,\,i+1}\,(\delta_i-\delta_{i+1})^2$$

When

$$n_{i,\,i+1}=n$$

and

$$N=nt$$

$$\lambda_s=t/4\sum_{i=1}^{t}(\delta_i-\delta_{i+1})^2$$

$$=t/2\left[\sum_{i=1}^{t}\delta_i^2-\sum_{i=1}^{t}\delta_i\,\delta_{i+1}\right] \quad \text{.. (12)}$$

If we could consider alternatives of the type

$$\pi_i = \frac{1}{t} + \frac{\gamma_{in}}{\sqrt{n}}$$

where

$$\lim_{n \to \alpha} \gamma_{in} = \gamma_i$$

so that

$$\gamma_i = t^{-1/2}\, \delta_i$$

(12) can be put in the form

$$\lambda_s = \tfrac{1}{2}\left[\sum_{i=1}^{t} \gamma_i^2 - \sum_{i=1}^{t} \gamma_i\, \gamma_{i+1}\right],$$

a result which we can get, if we derive the expression from first principles. Hence $T(^1)$ given in (7) has a non-central χ^2 distribution with $(t-1)$ degrees of freedom and non-centrality parameter λ_s. The limiting distribution of $T(^1)$ under H_1 is given by

$$f(T^{(1)}) = \frac{e^{-1/2\, T(1)}\, e^{-1/2\, \lambda_s}}{2^{1/2\,(t-1)}} \sum_{h=0}^{\alpha} \frac{(T^{(1)})^{1/2\,(t-1)+h-1}\, \lambda_s h}{\Gamma[\frac{1}{2}(t-1)+h]2^{2h}\, h!}$$

Then the power of the BT model for symmetrical pairs is given by

$$B\,(\lambda_s/\alpha,\ t-1,\ \infty) = \int_{t-1}^{\infty} f(T^{(1)})\, dT^{(1)}$$

where $\chi^2_{\alpha,\, t-1}$ is the α-level significance value of a central χ^2 distribution with $(t-1)$ *d.f.* Hence the powers ean be directly read from the charts of Pearson and Hartley (1951).

2.5. Generalised Asymptotic Relative Efficiency

Davidson and Bradley (1970) have adapted to the multivariate situation the general asymptotic efficiency of test procedures defined by Hoeffding and Rosenblatt (1955) and Noether (1957). The theory of the test procedure is as follows. Let the test parameter be

$$\theta = \theta^N = \theta^0 + dN^{-r} \qquad \text{...(14)}$$

where θ° is the value of θ under null-hypothesis. Let the test statistic T under H_0 have the limiting c.d.f. $H_0\,(t)$ and under the alternative (13) the limiting c.d.f. $H_1\ (t,\ d)$. Let t be such that $H_0\,(t_\alpha) = 1-\alpha$ and $D(\alpha,\ \beta)$ be the value of d for which $H_1\,[t_\alpha,\ D(\alpha,\ \beta)] = 1-\beta$ where α and β are respectively the type I and type II error rates. Given two tests T_1 and T_2, the asymptotic efficiency of T_1 and T_2 is defined as

$$E_{T_1T_2}(\alpha, \beta)=[D_2(\alpha, \beta)/D_1(\alpha, \beta)]^{-\frac{1}{r}} \quad ...(15)$$

Applying this to the comparison of symmetrical pairs to full pairs we find that $D_1(\alpha, \beta)$ for symmetrical pairs is such that

$$H_1[t_\alpha, D_1(\alpha, \beta)]=1-\beta.$$

In our test situation

$$d_{in}=\pi_i-\frac{1}{t}$$

and $\quad d_{in}\rightarrow 0 \quad$ as $\quad n\rightarrow\infty$

Hence $\quad \theta^\circ=0.$

Thus $\quad \theta^N=\theta^\circ+\frac{d}{N}=\frac{\lambda'}{N}$

when $\quad r=1.$

Hence $D_1(\alpha, \beta)$ is the value of λ' corresponding to α and β, from the limiting distribution of T'. Similarly $D_2(\alpha, \beta)$ can be obtained from the limiting distribution of T. Hence the efficiency of T' to T is

$$E_{T'T}=\frac{\lambda(t-1, \alpha, \beta)}{\lambda'(t-1, \alpha, \beta)}$$

Thus both tests are asymptotically equally efficient. Comparing symmetrical pairs T' with the corresponding multi-binominal procedure T_s', it is easily seen that

$$E_{T'T_s'}=\frac{\lambda'_s(t, \alpha, \beta)}{\lambda'(t-1, \alpha, \beta)}$$

where λ', λ'_s are the non-centrality parameters of the corresponding limiting distributious of T' and T_s'. The values of this efficiency are computed for $t=2, 3, 4$ and $\alpha=\beta=\cdot 05$ and presented below

t	α	β	$E_{T'T'_s}$
2	·05	·05	1·167
3	·05	·05	1·129
4	·05	·05	1·075

2·6. A Test for Appropriateness of the BT Model for Symmetrical Pairs

With unequal number of observations per pair, the test statistic becomes

$$\chi^2=-2ln\lambda$$

$$=2ln10\Big(\sum_i a_{i,\,i+1} \log a_{i,\,i+1}-\sum_i n_{i,\,i+1} \log n_{i,\,i+1}+B^{(1)}$$

the null distribution for which is a central chi-square with one degree of freedom.

3. Bradley-Terry Model with Ties

3·1. **The Model.** A Bradley-Terry Model for symmetrical pairs with ties can be developed as follows :—

Let the treatments have true treatment ratings $\pi_1, \pi_2, \ldots \pi_t$ on a subjective continuum such that $\pi_i > 0$ $(i = 1, 2, \ldots, t)$ and $\sum_{i=1}^{t} \pi_i = 1$.

The probability that i is preferred to j when i and j are compared is

$$\pi_{ij} = \pi_i \, (\pi_i + \pi_j)^{-1} = \frac{1}{4} \int_{-(V_i - V_j)}^{\infty} \text{Sech}^2\,(y/2)\,dy \text{ where } V_i = ln\pi_i.$$

It is evident from this function that π_{ij} depends only on $(V_i - V_j)$ where $(V_i - V_j)$ is the difference between the true merits of treatment i and treatment j and the probability of judge preferring i to j is a monotonic increasing function of $(V_i - V_j)$. Let us consider d_{ij} as the judge's estimate of difference $(V_i - V_j)$. Then the distribution function of d_{ij} is assumed to be

$$P\,(d_{ij} > d) = \tfrac{1}{4} \int_{-(V_i - V_j) + d}^{\infty} \text{Sech}^2\,(y/2)\,dy$$

The judge will prefer i to j if d_{ij} is positive. Then

let $\theta = \text{Exp. } \eta$ and $\pi_i = \text{Exp. } V_i$ $(i = 1, 2, \ldots, t)$,

where η is a threshold parameter of sensory perception for the judge. From these the preference probabilities will be

$$\pi_{i \cdot ij} = \pi_i \, (\pi_i + \theta \pi_j)^{-1}, \quad \pi_{j \cdot ij} = \pi_j (\theta \pi_i + \pi_j)^{-1} \text{ and}$$

$$\pi_{0 \cdot ij} = \frac{\pi_i \;\; \pi_j \, (\theta^2 - 1)}{(\pi_i + \theta \pi_j)(\theta_i \pi + \pi_j)}$$

3.2. **Estimation of θ and π_i's.** Using the formulations and notations as in Rao and Kupper (1967) the likelihood of the observed outcome of the experiment is propor tional to

$$L(\pi, \; \theta) = (\theta^2 - 1)^{N-n} \prod_{i}^{t} \pi_i^{\,b_i} \prod_{\substack{i=1 \\ j=i-1 \,\&\, i+1}}^{t} (\pi_i + \theta \pi_j)^{-b_{ij}}$$

and the estimation equations become

$$\left.\begin{aligned} &\frac{b_i}{p_i} - \sum_{\substack{j=i-1\ \& \\ i+1}} [(b_{ij}/(p_i+\hat{\theta}p_j)] \\ &-\sum_{\substack{j=i-1\ \& \\ i+1}} [b_{ji}\hat{\theta}/(\hat{\theta}p_i+p_j)]=0 \quad (i=1, 2, \ldots, t) \\ \text{and} \quad &\sum_{i=1}^{i} p_i=1 \end{aligned}\right\} \quad \ldots(15)$$

$$\text{and } 2(\text{N}-n)\hat{\theta}/(\hat{\theta}^2-1) - \sum_{i=1}^{t} \sum_{\substack{j=i-1\ \& \\ i+1}} (b_{ij}p_j/(\hat{\theta}p_j+p_i)=0 \quad \ldots(16)$$

This system of equations can be solved for $\hat{\theta}$ and p_i $(i=1, .., 2, .., t)$ by an iterative procedure.

3·3. **Tests of Hypotheses.** The two large sample tests of hypotheses to test for the treatment ratings p_i and threshold parameter θ give expressions for test statistics which can be obtained from adjustments for summation from Rao and Kupper (1967).

Test 3. The adequacy of the model can be tested by

$$\chi_3^2 = \sum_{i=1}^{t} \left[\frac{(n_{i\cdot\, i,\, j-1} - r_{i,\, i-1}\, p_{i\cdot\, i,\, i-1})^2}{r_{i,\, i-1}\, p_{i\cdot\, i,\, i-1}} + \frac{(n_{0\cdot\, i,\, i-1} - r_{i,\, i-1}\, p_{0\cdot\, i,\, i-1})^2}{r_{i,\, i-1}\, p_{0\cdot\, i,\, i-1}} + \frac{(n_{i-1\cdot\, i,\, t-1} - r_{i,\, i-1}\, p_{i-1\cdot\, i,\, i-1})^2}{r_{i,\, i-1}\, p_{i-1\cdot\, i,\, i-1}} \right]$$

where $p_{0\cdot\, i,\, i-1}, p_{i\cdot\, i,\, i-1}$ and $p_{i-1\cdot\, i,\, i-1}$

are the estimates for $\pi_{0\cdot\, i\cdot\, i-1}\cdot \pi_{i\cdot\, i,\, i-1}$ and $\pi_{i-1\cdot\, i,\, i-1}$ respectively. For large samples χ_3^2 will have an approximate χ^2 distribution with t degrees of freedom.

4. Illustrative Examples

(*i*) In an experiment for comparing four improved varieties of wheat for texture using symmetrical pairs with $n=2$, the score vector was found to be (3, 0, 4,1) for the varieties labelled 1 to 4. Then for the test under case (*i*) of the BT Model without ties the ratings vector as obtained from the normal equations is (·325, .10,

·45, ·125) with a probability ·0312 (from appendix). Thus under the null hypotheses the probability of obtaining such a score vector is very small. Thus the varieties differ significantly among themselves for texture. Under the general hypotheses and case (2) we can only perform the asymptotic tests because the B and $B^{(2)}$ functions have not been tabled. Under case (2) for the score vector (3, 0, 4, 1), we test the hypothesis that $p_1=p_2=p$ and $p_3=p_4=p'$. Then $B_1=1$ and $B_2=3$; $p=·125$; $p'=·375$ from (8) and (9). Hence $B^{(2)}$ from (11) equals 2·1810 and $T^{(2)}=6·0684$ which at one degree of freedom and ·05 level is significant, Thus the groups differ significantly.

Consider the general hypothesis

$$H_a : \pi_1=\pi_2=\pi_{(2)} \; ; \; \pi_3=\pi_{(2)} \; ; \; \pi_4=\pi_{(3)}.$$

Then the score vector under the modified hypothesis in the above example becomes (3, 4, 1). Using equation (2) the ratings by an iterative procedure, work out approximately as $p_{(1)}=·32$, $p_{(2)}=·17$; $p_{(3)}=·19$. Then from (3) $B=·7839$ and $T=7·4804$. For 2 degrees of freedom and ·05 level of probability this T is significant showing that the groups differ significantly among themselves.

(*ii*) For the illustration of the model with ties we use the data of an experiment on palatability testing conducted in the Quality Testing Laboratory of the Indian Agricultural Research Institute, New Delhi. Five improved varieties of wheat namely 1. Kalyan Sona, 2. Sonalika, 3. Choti Lerma, 4. Sharbati Sonora, and 5. N.P. 718 were used for the test. Five judges were selected by the duo-trio test. The symmetrical pairs were randomised and presented in random order to each judge. The experiment was replicated thrice. The data pooled over the judges are given below:

i	*Pair*	$n_{i\cdot ij}$	$n_{0\cdot ij}$	$n_{j\cdot ij}$	*Total*
1.	(1,2)	5	3	7	15
2.	(2,3)	6	4	5	15
3.	(3,4)	6	3	6	15
4.	(4,5)	4	3	8	15
5.	(5,1)	6	2	7	15

Here N, the total numbers of trials =75 and n, the number of trials which gave no tie =60. An approximate set of solutions is given by $\theta=·63$; $p_1=·13$; $p_2=·25$; $p_3=·15$; $p_4=·08$ and $p_5=·39$.

Then to test the hypothesis $H_0 : \pi_i=1/5$ against the alternative H_1 : $\pi_i \neq \pi_j$, the test statistic from [9] viz $\chi_1^2=86·67$ which is highly significant at 4 degrees of freedom. Thus the varieties differ significantly in palatability.

To test for the threshold parameter $\theta = \cdot 5$ against the alternative $\theta \neq \cdot 5$, we get a new set of solutions of p_i's under the hypothesis that $\theta = \cdot 5$ as $p_1' = \cdot 18$; $p'_2 = \cdot 20$; $p_3' = \cdot 21$; $p_4' = \cdot 23$; $p_5' = \cdot 18$. Thus χ_2^2 from [4], works out to be 32·382 which is significant for 1 degree of freedom. Hence the threshold parameter is significantly different from ·5.

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APPENDIX

Tables for Symmetrical Pairs

$t=4,\ b=4,\ k=2,\ r=2,\ n=1$

(t=number of objects ; b=number of pairs ; k=plot size, r=number of replications of an object ; n=number of repetitions per pair).

a_1	a_2	a_3	a_4	p_1	p_2	p_3	p_4	$B(1)$	P
2	0	2	0	·5	0	·5	0	0	·125
2	1	0	1	·25	·25	·25	·25	1·20412	·875
1	1	1	1	·25	·25	·25	·25	1·20412	1·000

$t=4,\ b=8,\ k=2,\ r=4,\ n=2$

a_1	a_2	a_3	a_4	p_1	p_2	p_3	p_4	$B(1)$	p
4	0	4	0	·5	0	·5	0	0	·0078
3	3	0	2	·32	·32	·11	·25	·88098	·1641
4	2	0	2	·50	·20	·10	·20	1·28888	·2110
3	0	4	1	·325	·10	·45	·125	1·32136	·2422
4	1	1	2	·5	·125	·125	·25	1·50722	·3985
3	2	1	2	·40	·25	·10	·25	1·65801	·8048
3	1	1	3	·375	·125	·125	·375	1·71086	·9298
2	2	2	2	·25	·25	·25	·25	2·40824	1·0000

$t=5,\ b=5,\ k=2,\ r=2,\ n=1$

a_1	a_2	a_3	a_4	a_5	p_1	p_2	p_3	p_4	p_5	$B(1)$	p
1	1	1	1	1	·2	·2	·2	·2	·2	·50515	·0625
2	1	1	2	0	·2	·2	·2	·2	·2	1·5051	·6875
2	0	2	0	1	·4	·1	·2	·2	·2	6·3250	1·0000